Hence, $DE = a\sqrt{3}$. In the same manner we find $EF = b\sqrt{3}$ and $FD = c\sqrt{3}$.

$$\text{Area of triangle } DAE = \frac{a^2}{4}\sqrt{3}; \tag{2}$$

$$\text{area of triangle } EBF = \frac{b^2}{4}\sqrt{3}; \tag{3}$$

$$\text{area of triangle } FCD = \frac{c^2}{4}\sqrt{3}; \tag{4}$$

and

$$\text{area of triangle } DEF = \tfrac{3}{4}\sqrt{(a+b+c)(b+c-a)(a+c-b)(a+b-c)}. \tag{5}$$

$$\begin{aligned}(2) + (3) + (4) + (5) &= \text{the area of the polygon } AEBFCD\\ &= 2 \times \text{area of triangle } ABC.\end{aligned}$$

Equating the two expressions for the area of the triangle, we have

$$\frac{x^2}{4}\sqrt{3} = \tfrac{1}{8}(a^2+b^2+c^2)\sqrt{3} + \tfrac{3}{8}\sqrt{(a+b+c)(b+c-a)(a+c-b)(a+b-c)};$$

$$\therefore\quad x = \sqrt{\tfrac{1}{2}\{(a^2+b^2+c^2) + \sqrt{3}\,[(a+b+c)(b+c-a)(a+c-b)(a+b-c)]\}}.$$

In the triangles we have all the sides given, to find the angles.

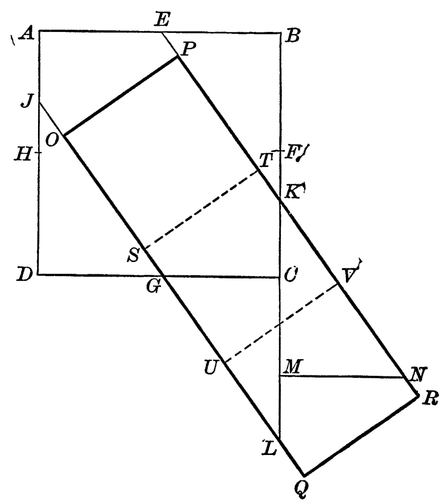
Also solved by NATHAN ALTSHILLER, NELLE L. INGELS and the PROPOSER.

515 (Geometry). Proposed by C. F. GUMMER, Kingston, Ontario.

Show how to cut up a square carpet and make it into three equal square carpets. Estimate the total length of seam in comparison with a side of the original carpet.

I. SOLUTION BY HARRY C. BRADLEY, Massachusetts Institute of Technology.

Construction. Let $ABCD$ be the original square. Take E, F, G, H as the middle points of its four sides. Lay off $BK = DJ$ equal to the distance $EF = GH$. Draw EK and JG, which are obviously parallel. Remove the triangles EBK and JDG, and place them in the positions



GCL and KMN, respectively. In any convenient position draw the line OP perpendicular to JG and EK. Remove the piece $AEPOJ$, and fit it into the position $MNRQL$. The square has now been transformed into the rectangle $OPRQ$, which is three times as long as wide, and may be divided by the lines ST and UV into three equal squares.

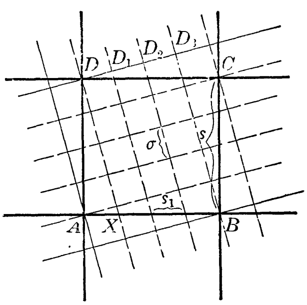
Proof. The area of the rectangle $OPRQ$ is evidently the same as that of the original square, and it is only necessary to prove that the long side, OQ, is three times the short side OP. The side $OQ = JL = JG + GL = JG + EK = 2JG$. To avoid fractions, let the side of the original square equal 6. Then $DG = DH = 3$, $JD = GH$ (by construction) $= 3\sqrt{2}$, and

$$JG = \sqrt{DG^2 + JD^2} = 3\sqrt{3}.$$

Whence, $OQ = 6\sqrt{3}$. The area of the rectangle $OPRQ$ = the area of the square $ABCD = 6 \times 6 = 36$. That is, $OP \times OQ = 36$, or $OP = 2\sqrt{3}$. Hence $OQ = 3OP$.

The Length of Seam. The length of seam, $KL + GC + MN$, is just twice the length of a side of the original square. For $GC + MN$ is the length of a side of the square, since each is equal to half a side, while $KL = KC + CL = KC + BK = BC$.

II. Solution by A. F. Frumveller, Marquette University.



Before attacking this special case, let us deduce the formulas that govern the partition of a square into N squares. Let $ABCD$ be a given square, s its side, and n a number such that $s = ns_1$; in the figure $n = 4$. Draw DX, and complete the figure as indicated. Then we have

$$(1)\quad \overline{DX}^2 = s^2 + \frac{s^2}{n^2} = \frac{s^2}{n^2}(n^2+1),$$

$$(2)\quad \overline{DX} = n\sigma + \frac{\sigma}{n} = \frac{\sigma}{n}(n^2+1);\quad \text{hence}\quad s^2 = (n^2+1)\sigma^2 = N\sigma^2.$$

This gives us the relation between the number of sub-squares, and the length of the fundamental segment on AB which produces this partition; $N = n^2 + 1$. The number of complete sub-squares lying within the given square is evidently the greatest integer in

$$\frac{(n-1)(n^2-n+1)}{n},$$

which number is the product of

$$\left(\frac{\overline{AB}}{s_1} - 1\right)\left(\frac{\overline{DX}}{\sigma} - 1\right);$$

the shorter formula conveyed by our figure, $(n^2+1) - 4n = (n-1)^2$, holds only when n is integral.

The number N is an integer not only when n is integral, but likewise when $n = \sqrt{k}$, in which case $N = k + 1$, and

$$s_1 = \frac{\sqrt{k}}{k}s;$$

if $N = 3$, $n = \sqrt{2}$, and

$$s_1 = \frac{s}{2}\sqrt{2},$$

or half the diagonal of the original square. The length of all the cuts made in the given square, when n is rational, is clearly $2n$ times $\overline{DX}$, or

$$2s\sqrt{n^2+1};$$

when n is irrational, it will be noted that the last cut parallel to $\overline{DX}$ within the square (call it $\overline{E_rX_r}$) does not reach the baseline, but crosses the side of the square at a point X_r. By similar triangles, we then have

$$\frac{\overline{DX}}{s_1} = \frac{\overline{E_rX_r}}{s - [ns_1]},$$

where the bracket means as usual, "the greatest integer in ns_1"; hence

$$\overline{E_rX_r} = \overline{DX}\left(\frac{s}{s_1} - \frac{[ns_1]}{s_1}\right) = \overline{DX}(n - [n]),$$

and

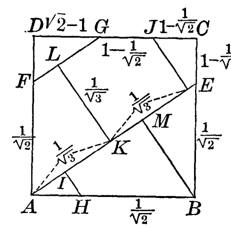
$$\Sigma(\overline{D_iX_i}) = \overline{DX}([n] + (n - [n])) = n \cdot \overline{DX}.$$

Since the construction of the cross-lines is similar, the sum of all the cuts is as before $2s\sqrt{n^2+1}$.

The skew lattice-work of our figure can be readjusted by slipping downwards each of the columns lying between D and C, until the points $D_1, D_2, \cdots, D_r$ rest on $\overline{DC}$; this is optional for n integral; but for n irrational it is *necessary*, in order to have congruent fragments along the sides of the given square, out of which our N sub-squares may be patched up. Thus we find the actual lines on which the original square must be partitioned. (This problem, for $N = 3$, is solved as above in Sundara Row's "Geom. Exercises in Paper Folding," but without any hint as to how the solution was arrived at.)

The patched squares have "seams" across them, whose total length is $2s$, since two equal segments from the sides combine to form one seam.

III. Solution by the Proposer.



Join AF to BE, GJ to AH, HB to JC and DG, CE to DF.

$$\text{Total seam} = AF + AH + HB + CE$$

$$= \left(\frac{1}{\sqrt{2}} + 1 + 1 - \frac{1}{\sqrt{2}}\right) AB = 2AB.$$

Also solved by E. B. Escott.

426 (Calculus). Proposed by C. N. SCHMALL, New York City.

If A be the area of a plane triangle constructed with the sides a, b, c, such that

$$a^3 + b^3 + c^3 = 3k^3,$$

show that the maximum value of A is $\frac{1}{4}k^2$.

Solution by Elijah Swift, University of Vermont.

Using the Heronian formula for the area of a triangle in terms of the sides, we have

$$16A^2 = -a^4 - b^4 - c^4 + 2a^2b^2 + 2b^2c^2 + 2c^2a^2.$$

Clearly we may make $16A^2$ a maximum instead of A.

By Euler's method of solving an isoperimetric problem, we must equate to zero the partial derivatives with respect to a, b, c, of $16A^2 - 4\lambda(a^3 + b^3 + c^3 - 3k^3)$. (For convenience I use -4λ instead of λ.) In this way we get the equation

$$-4a^3 + 4ab^2 + 4ac^2 - 4\lambda(3a^2) = 0,$$

and two more similar to it. Dividing by $4a$, which cannot be zero,

$$(1) \qquad a^2 + 3\lambda a = b^2 + c^2$$

and two equations, (2) and (3), obtained by permuting cyclically the letters a, b, c in (1).

But from (1) a must satisfy the equation

$$(4) \qquad 2x^2 + 3\lambda x = a^2 + b^2 + c^2,$$

and (2) and (3) show that b and c are roots of the same equation. Since this equation has only one positive root, a, b, c must all be equal, and the triangle equilateral, whence it appears that its area must be $\frac{1}{4}k^2\sqrt{3}$, which is not the value given in the statement.

The same method used here shows that this result holds when the auxiliary condition is $a^n + b^n + c^n = 3k^n$, $n > 1$.

It is evident from the conditions of the problem that the values found give a maximum.

Also solved by R. A. Johnson, S. A. Corey, and J. B. Reynolds.